Internal stabilization of transport systems

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From controllability to stabilization.

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Controllable if

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$$\varphi \in H^{m-1}_{per} \cap H^m_{(pw)} \quad (m \geq 1)$$

Results

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Theorem (Finite-time stabilization in Sobolev norms)

Under the same conditions, there exists a feedback law that stabilizes the system in finite time $T = L$.

Stabilization of hyperbolic systems

Approaches to solve a stabilization problem:

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  Fredholm transformations: Kuramoto-Shivashiinski (Coron-Lu), KdV (Coron-Lu), Schrödinger (Coron et al.), Transport (today).

Summary

1 Introduction

2 From controllability to stabilization
- Pole-shifting in finite dimension
- Strategy of proof for the transport equation

Classical pole-shifting

Consider the finite-dimensional **controllable** control system

$$\dot{x} = Ax + Bu(t), \quad x \in \mathbb{C}^n, A \in \mathcal{M}_n(\mathbb{C}), B \in \mathcal{M}_{n,1}(\mathbb{C}).$$

Kalman condition: $rank\{A^n B \mid n = 0, \cdots, n-1\} = n$.

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Poleshifting: $\forall P,\ \exists K \in \mathcal{M}_{1,n}(\mathbb{C}), \quad \chi(A+BK) = P$.

Finite-dimensional backstepping

Another way of shifting poles: map

$$\dot{x} = Ax + B(Kx + v(t))$$

into the stable system

$$\dot{x} = (A - \lambda I)x + Bv(t).$$

The mapping T should be invertible and satisfy

$$T(A + BK) = AT - \lambda T,$$
$$TB = B.$$

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“Backstepping equations”

Finite-dimensional backstepping

Proposition

If the system (6) ***is controllable***, *then there exists a unique pair* (T, K) *satisfying conditions* (7)

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Controllability $\rightarrow$ **basis property**

Finite-dimensional backstepping

$$(A - \lambda I)T - TA = TBK,$$

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\begin{gathered}
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K is a parameter of T.

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Our system

Linear feedbacks:

$$\langle \alpha(t), F \rangle = \sum_{n \in \mathbb{Z}} \overline{F_n} \alpha_n(t) = \int_0^L \bar{F}(s) \alpha(s) ds$$

Closed-loop system:

$$\begin{cases} \alpha_t + \alpha_x = \langle \alpha(t), F \rangle \varphi(x), \ \ x \in [0, L], \\ \alpha(t, 0) = \alpha(t, L), \ \ \forall t \geq 0. \end{cases}$$

Target system:

$$\begin{cases} z_t + z_x + \lambda z = 0, \quad x \in (0, L), \\ z(t, 0) = z(t, L), \quad t \geq 0. \end{cases}$$

Kernel equations

T is a kernel operator: $f \mapsto \int_0^L k(x,y)f(y)dy$.

Operator equation $\xrightarrow{\text{Formal computations (IBP...)}}$ PDE for $k(x,y)$.

$(A-\lambda I)T - TA = TBK$	$\begin{cases} k_x + k_y + \lambda k + \int_0^L k(x,s)\varphi(s)ds\bar{K}(y) = 0, \\ k(0,y) = k(L,y), \\ k(x,0) = k(x,L). \end{cases}$

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Controllability gives a basis property!

Invertibility and feedback

$$T\alpha = \sum_{n \in \mathbb{Z}} \alpha_n T e_n, \quad \alpha \in H^m_{per}$$

Invertible iff $|K_n| \sim n^m$ ($n^m \alpha_n \in \ell^2$).

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$TB = B \to b_i(Ke_i) = \tilde{b}_i.$

Controllability:

$b_i \neq 0 \to Ke_i = \dfrac{\tilde{b}_i}{b_i}$

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But...$\varphi \notin H^m_{per}$. $T\varphi$?

Weak condition:

$\varphi^{(N)} \xrightarrow[N\to\infty]{H^{m-1}_{per}} \varphi, \quad T\varphi^{(N)} \rightharpoonup \varphi$

iff $K_n := -\dfrac{2}{L\overline{\varphi_n}} \dfrac{1-e^{-\lambda L}}{1+e^{-\lambda L}} \sim n^m$

Dirichlet convergence theorem

Almost done...

- **Kernel equations** Derived formally using the $TB = B$ condition!

$$\begin{cases} \textbf{Basis property} \\ \textbf{Definition of } (T, K) \\ \textbf{Invertibility of } T \end{cases} \rightarrow \text{weak } TB = B!$$

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- **Operator equality** $T(A + BK) = AT - \lambda T$ on $D(A + BK)$.

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$$\left\{\begin{array}{l}\textbf{Basis property}\\ \textbf{Definition of } (T,K) \quad \to \text{weak } TB = B! \\ \textbf{Invertibility of } T\end{array}\right.$$

- **Operator equality** $T(A+BK) = AT - \lambda T$ on $D(A+BK)$.
- **Well-posedness of the closed-loop system.** Lumer-Phillips theorem (study the regularity of the feedback law).

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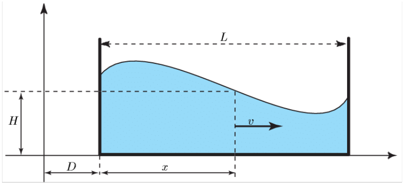
- Works thanks to **exact** controllability.

Next step

$$\begin{cases} H_t + (HV)_x = 0, \\ V_t + \left(gH + \dfrac{V^2}{2}\right)_x = \underbrace{-u(t)}_{\text{acceleration}}, \\ V(t,0) = V(t,L) = 0, \quad \forall t \geq 0. \end{cases}$$



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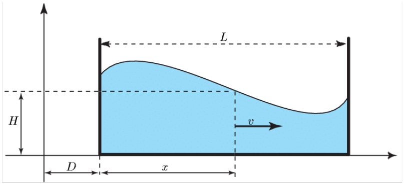

Linearised around $(H^\gamma, V^\gamma) := (H_0 - \gamma x, 0)$ (constant acceleration):

$$\begin{cases} h_t + h^\gamma (V)_x = 0, \\ v_t + g\,(h)_x = -u(t), \\ v(t,0) = v(t,L) = 0, \quad \forall t \geq 0. \end{cases}$$

Controllable. Stabilizable?